# Lecture 2: Standard Game Background (inspired from the Deep Learning Book)

Start Recording!

# Outline of the lecture

1. Motivations: Simple two player games
2. Solutions concepts
    a. Nash Equilibria
    b. Correlated Equilibria
    c. Stackelberg Equilibria
3. Complexity of these solution concepts

# Disclaimer

- Game theory is a topic by itself
- Here: Quick introductions of some notions useful for the course
- (My very biased choice)

- Many more things to say about that topic!

  (you can read the algorithmic game theory book)

# Prisoners' Dilemma

| P1 \ P2 | Confess | Silent |
|---|---|---|
| Confess | 4, 4 | 1, 5 |
| Silent | 5, 1 | 2, 2 |

- A single equilibrium (total value = 8)
- Suboptimal (could achieve 4)

# Prisoners' Dilemma

Conclusion:

- If the agents do not **cooperate**, the best (global) outcome possible is missed.

# COP 21 Game

- N governments
- 2 actions per states
    - Do not pollute (Cost = 3)
    - Pollute (cost = 1 and +1 to everyone)

<u>Exercice:</u> What is the Equilibrium? What is its value? What could we achieve with cooperation?

# COP 21 Game

Conclusion:

- Global warming cannot be avoided.

# COP 21 Game

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~~Global warming cannot be avoided.~~

# COP 21 Game

Conclusion:

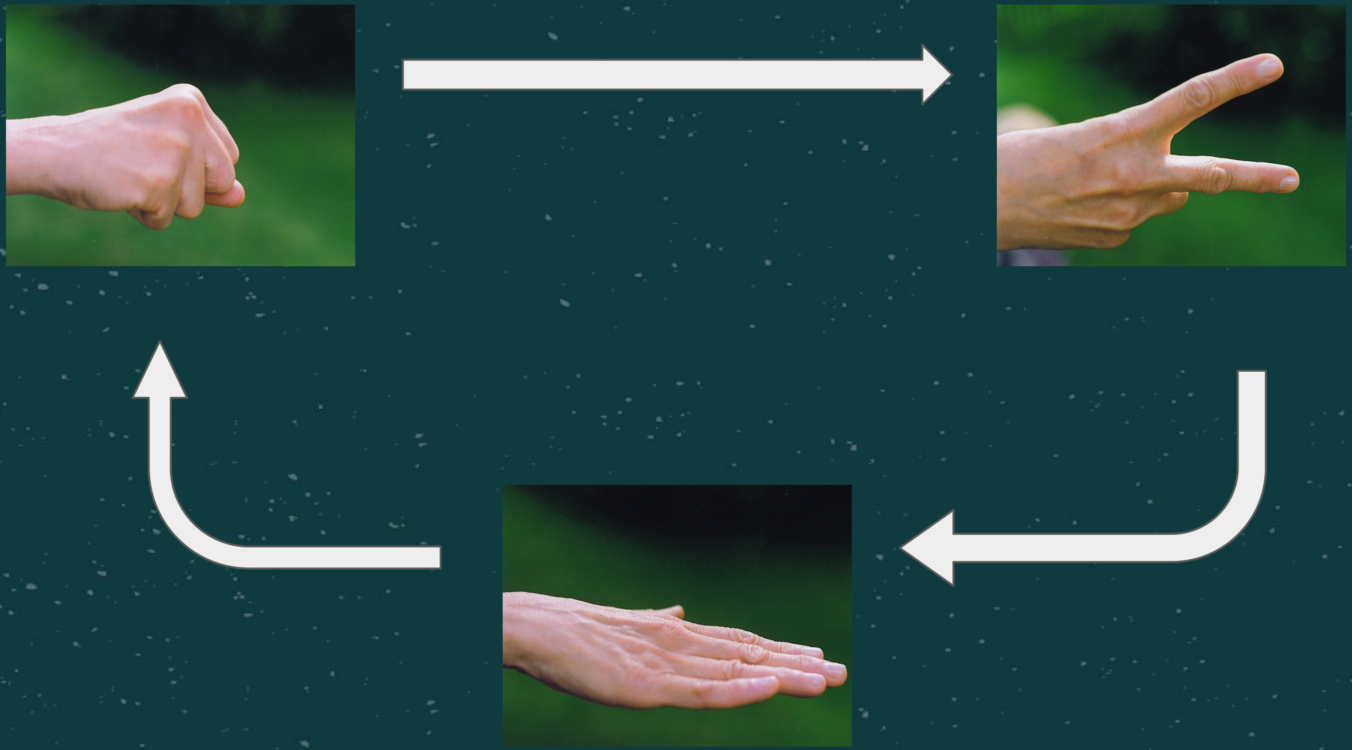
- If we only think about our **local problems**, global warming cannot be avoided.

# COP 21 Game

Conclusion:

- If we only think about our **local problems**, global warming cannot be avoided.
- The two games considered have pure **stable** solutions.

# Rock Paper Scissors



# Rock Paper Scissors

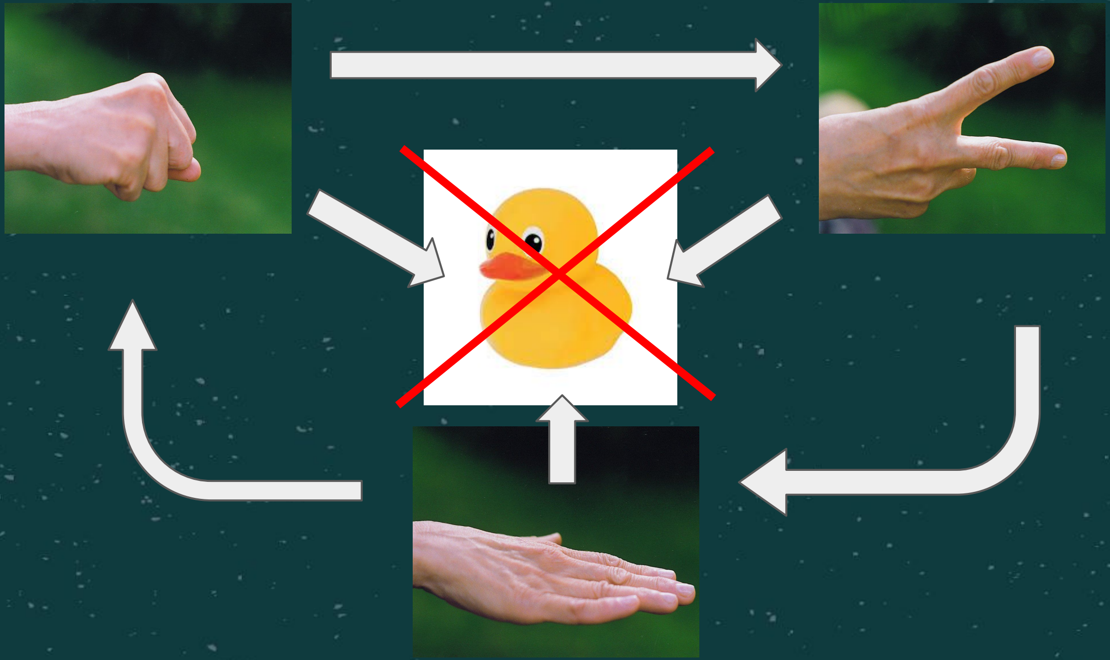
Conclusion:

- There is no **best single** strategy.

- But there is a **best distribution.**

- The best distribution may have a small support.

# Rock Paper Scissors



# Multi-player Game

- Simultaneous move games.
- n players, each player pick a strat and occurs a loss

$$\ell_k(s_1, \ldots, s_n) = \ell_k(\boxed{s_k}, s_{-k})$$

- Goal of the player: *minimize* their loss.

# Zero-Sum Two-Player Games

- Zero-sum: $\sum_{k=1}^{n} \ell_k = 0$
- Two player: n=2
- Simplification: $i \in [n] = \{1, \ldots, n\}, \; j \in [m]$

Action of player 1

Action of player 2

- Game: $$\min_{i \in [n]} \max_{j \in [m]} \ell_{ij}$$

# Zero-Sum Two-Player Games

- RPS example: need to mix strategies.

$$p = [p_1, \dots, p_n] \in \Delta_n \,, \qquad q = [q_1, \dots, q_m] \in \Delta_m$$

Proba over the strategies

$$\Delta_n := \{p \in \mathbb{R}^n \;:\; p_1 + \dots p_n = 1 \,,\, p_i \geq 0\}$$

Payoff: $\ell(p, q) := \mathbb{E}_{i \sim p \,,\, j \sim q}[\ell_{ij}] = p^\top L q$

$$\min_{p \in \Delta_n} \max_{q \in \Delta_m} p^\top L q$$

# Zero-Sum Two-Player Games

- RPS example: need to mix strategies.

Exercice: What is the matrix L for the rock-paper-scissor game?

- Payoff:

$$\ell(p,q) := \mathbb{E}_{i\sim p\,,\,j\sim q}[\ell_{ij}] = p^\top Lq$$

$$\min_{p\in\Delta_n}\max_{q\in\Delta_m} p^\top Lq$$

# First solution concept: Nash Equilibrium of a Game

- Best **worst-case** move.

$$s^* \in \mathrm{Nash} \iff \ell_k(s_k^*, s_{-k}^*) \leq \ell_k(s_k, s_{-k}^*) \, \forall s$$

- Zero-sum two-player case:

<u>Pure:</u> $$\ell_{i^*j} \leq \ell_{i^*j^*} \leq \ell_{ij^*} \quad \forall i \in [n], j \in [m]$$

<u>Mixed:</u> $$p_*^\top L q \leq p_*^\top L q_* \leq p^\top L q_* \quad \forall p \in \Delta_n, q \in \Delta_m$$

# First solution concept: Nash Equilibrium of a Game

- Best **worst-case** move.

$$s^* \in \mathrm{Nash} \iff \ell_k(s_k^*, s_{-k}^*) \leq \ell_k(s_k, s_{-k}^*) \, \forall s$$

- Zero-sum two-player case:

<u>Pure:</u> $\ell_{i^*j} \leq \ell_{i^*j^*} \leq \ell_{ij^*}$

No Equilibria **in General!**

<u>Mixed:</u> $p_*^\top L q \leq p_*^\top L q_* \leq p^\top L q_* \quad \forall p \in \Delta_n \, , q \in \Delta_m$

# First solution concept: Nash Equilibrium of a Game

- Best **worst-case** move.

$$s^* \in \text{Nash} \iff \ell_k(s_k^*, s_{-k}^*) \leq \ell_k(s_k, s_{-k}^*)\, \forall s$$

- Zero-sum two-player case:

Pure: $\ell_{i^*j} \leq \ell_{i^*j^*} \leq \ell_{ij^*}$

No Equilibria **in General!**

Mixed: $p_*^\top L q \leq p_*^\top L q_* \leq p^\top L q_*$

Equilibria **Always exists**

# First solution concept: Nash Equilibrium of a Game

**Theorem 1.8** *Any game with a finite set of players and finite set of strategies has a Nash equilibrium of mixed strategies.*

- Will cover the zero-sum two-player case (proof)
- <u>Exercice:</u> Show that for any compact sets X,Y and continuous function f we have:

$$\max_{y \in Y} \min_{x \in X} f(x, y) \leq \min_{x \in X} \max_{y \in Y} f(x, y)$$

# Nash Equilibrium

**Exercices:**

- Give an algorithm for finding a Nash equilibrium for a two player game (non-zero sum but with finite number of strategies).
- Give its complexity.

# Second solution concept: Correlated Equilibrium

- Players can coordinate using a recommendation from a "third party".
- The recommender is $p \in \Delta_{nm}$ where $p_{ij}$

Proba of recommending i for player 1 and j for player 2

- Equilibrium: point where I have no incentive not to follow the recommendation.

# Second solution concept: Correlated Equilibrium

- Equilibrium: point where I have no incentive not to follow the recommendation.

$$\sum_{j=1}^{m} p_{ij}\ell_{ij}^{(1)} \leq \sum_{j=1}^{m} p_{ij}\ell_{i'j}^{(1)} \qquad \forall i, i' \in [n]$$

- Same for player 2.

# Second solution concept: Correlated Equilibrium

## Example: Traffic game

| 1 \ 2 | Cross | Stop |
|---|---|---|
| Cross | -100<br>-100 | 0<br>1 |
| Stop | 1<br>0 | 0<br>0 |

# Second solution concept: Correlated Equilibrium

Example: Traffic game

| 1 \ 2 | Cross | Stop |
|---|---|---|
| Cross | -100, -100 | 1, 0 |
| Stop | 0, 1 | 0, 0 |

If the agents do not **cooperate**, one agent cannot cross.

# Second solution concept: Correlated Equilibrium

## Example: Traffic game

Exercice: Find the Nashes of this game.
Find the correlated equilibria of this game.

| 1 \ 2 | Cross | Stop |
|---|---|---|
| Cross | -100<br>-100 | 0<br>1 |
| Stop | 1<br>0 | 0<br>0 |

# Second solution concept: Correlated Equilibrium

- Nash Equilibrium: $p \in \Delta_n$ , $q \in \Delta_m$
- Correlated Eq: Players follow $p_{rec} \in \Delta_{nm}$

$$\Delta_n \times \Delta_m \quad \text{v.s.} \quad \Delta_{nm}$$

Much larger space

# Complexity of Nash in One Slide

- The proofs of these results would take a whole course.
- Zero-sum two-players ⟷ Bilinear minimax ⟷ linear program.
- Non-zero sum: PPAD hard
  - [Daskalakis et al. 2006] for three player games. (conf paper from before 2005)
  - Extended for non-zero-sum two-player games by Chen and Deng [2005b].
- PPAD notion of hardness different from NP-hard.

Costis Daskalakis

See algorithmic Game theory book for more details.

# Complexity of Correlated Equilibria

- Problem find: $p \in \Delta_{nm}$
    - Such that $\sum_{j=1}^{m} p_{ij} \ell_{ij}^{(1)} \leq \sum_{j=1}^{m} p_{ij} \ell_{i'j}^{(1)} \quad \forall i \in [n]$ and $\sum_{j=1}^{m} p_{ij} \ell_{ij}^{(2)} \leq \sum_{j=1}^{m} p_{ij} \ell_{ij'}^{(2)} \quad \forall j, j' \in [n]$
- Linear constraints on $p \in \Delta_{nm}$

| 1 \ 2 | Cross | Stop |
|---|---|---|
| Cross | -100 / -100 | 0 / 1 |
| Stop | 1 / 0 | 0 / 0 |

Source: algorithmic game theory book

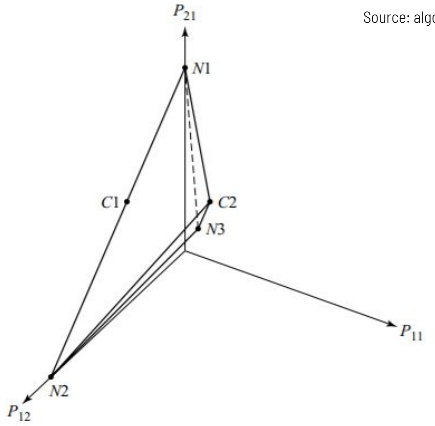



**Figure 2.4.** The three NASH equilibria (N1, N2, N3) of the drivers' game are vertices of the polytope of the correlated equilibria. Two other correlated equilibra are shown (C1, C2).

# Nash Equilibrium Vs. Correlated Equilibrium

$$p \in \Delta_n \,,\; q \in \Delta_m$$

$$\sum_{j=1}^{m} p_i q_j \ell_{ij}^{(1)} \leq \sum_{j=1}^{m} p_i q_j \ell_{i'j}^{(1)} \quad \forall i, i' \in [n]$$

Bilinear problem (deg = 2)

**PPAD** (except zero-sum two player)
(basically means not tractable in practice )

$$p_{rec} \in \Delta_{nm}$$

$$\sum_{j=1}^{m} p_{ij} \ell_{ij}^{(1)} \leq \sum_{j=1}^{m} p_{ij} \ell_{i'j}^{(1)} \quad \forall i \in [n]$$

Linear problem!

Can use Linear problem solver
(basically means often solvable
in practice )