Frank-Wolfe Algorithms for Saddle Point problems



Gauthier Gidel[1] Tony Jebara[2] Simon Lacoste-Julien[3]

[1]INRIA Paris, Sierra Team [2]Department of CS, Columbia University

[3]Department of CS & OR (DIRO) Université de Montréal

10th December 2016

Overview

- Frank-Wolfe algorithm (FW) gained in popularity in the last couple of years.
- Main advantage: FW only needs LMO.
- Extend FW properties to solve saddle point problem.
- **Straightforward** extension but **Non trivial** analysis.

Saddle point and link with variational inequalities

Let $\mathcal{L} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, where $\mathcal{X}$ and $\mathcal{Y}$ are convex and compact.

$$\boxed{\text{Saddle point problem:} \quad \text{solve } \min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{y} \in \mathcal{Y}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y})}$$

A solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is called a ***Saddle Point***.

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- **Necessary *stationary conditions:***

$$\langle \boldsymbol{x} - \boldsymbol{x}^*, \quad \nabla_x \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{y}^*) \rangle \geq 0$$

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$$\begin{aligned} \langle \boldsymbol{x} - \boldsymbol{x}^*, \quad \nabla_x \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{y}^*) \rangle &\geq 0 \\ \langle \boldsymbol{y} - \boldsymbol{y}^*, -\nabla_y \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{y}^*) \rangle &\geq 0 \end{aligned}$$

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- ***Variational inequality:***

$$\forall \boldsymbol{z} \in \mathcal{X} \times \mathcal{Y} \quad \langle \boldsymbol{z} - \boldsymbol{z}^*, g(\boldsymbol{z}^*) \rangle \geq 0$$

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- **Sufficient *condition***: ***Global solution*** if $\mathcal{L}$ *convex-concave.* $\forall (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$

$$\boldsymbol{x}' \mapsto \mathcal{L}(\boldsymbol{x}', \boldsymbol{y}) \text{ is convex} \quad \text{and} \quad \boldsymbol{y}' \mapsto \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}') \text{ is concave.}$$

Motivations: games and robust learning

- ***Zero-sum games with two players:***

$$\min_{x \in \Delta(I)} \max_{y \in \Delta(J)} \boldsymbol{x}^\top M \boldsymbol{y}$$

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- ***Generative Adversarial Network*** (GAN)
- ***Robust learning:***[1] We want to learn

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \ell\left(f_\theta(x_i), y_i\right) + \lambda \Omega(\theta)$$

with an uncertainty regarding the data:

$$\min_{\theta \in \Theta} \max_{w \in \Delta_n} \sum_{i=1}^{n} \omega_i \ell\left(f_\theta(x_i), y_i\right) + \lambda \Omega(\theta)$$

$\boxed{\text{Minimize the } \textbf{worst case} \rightarrow \text{gives } \textbf{robustness}}$

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Problem with Hard projection

The ***structured SVM:***

$$\min_{\omega \in \mathbb{R}^d} \lambda \Omega(\omega) + \frac{1}{n} \sum_{i=1}^{n} \underbrace{\max_{y \in \mathcal{Y}_i} \left(L_i(y) - \langle \omega, \phi_i(y) \rangle \right)}_{\text{structured hinge loss}}$$

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Regularization: penalized $\to$ constrained.

$$\min_{\Omega(\omega) \leq \beta} \max_{\alpha \in \Delta(|\mathcal{Y}|)} b^T \alpha - \omega^T M \alpha$$

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- ***Structured sparsity*** norm (group lasso norm).

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Hard to project when:

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- The output $\mathcal{Y}$ is structured: ***exponential*** size.

Standard approaches in literature

Simplest algorithm to solve Saddle point problems is the *projected gradient algorithm.*

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\begin{aligned}
\boldsymbol{x}^{(t+1)} &= P_{\mathcal{X}}(\boldsymbol{x}^{(t)} - \eta \nabla_x \mathcal{L}(\boldsymbol{x}^{(t)}, \boldsymbol{y}^{(t)})) \\
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\end{aligned}
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For non-smooth optimization,

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\frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{x}^{(t)}, \boldsymbol{y}^{(t)} \right) \underset{T \to \infty}{\longrightarrow} (\boldsymbol{x}^*, \boldsymbol{y}^*)
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Faster algorithm: *projected extra-gradient algorithm*.

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Faster algorithm: *projected extra-gradient algorithm*.

Can use LMO to compute approximate projections[2].

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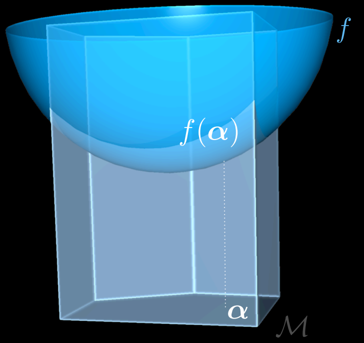
The FW algorithm

Algorithm Frank-Wolfe algorithm

Let $\boldsymbol{x}^{(0)} \in \mathcal{X}$
for $t = 0 \dots T$ **do**
  Compute $\boldsymbol{r}^{(t)} = \nabla f(\boldsymbol{x}^{(t)})$
  Compute $\boldsymbol{s}^{(t)} \in \underset{\boldsymbol{s} \in \mathcal{X}}{\operatorname{argmin}} \left\langle \boldsymbol{s}, \boldsymbol{r}^{(t)} \right\rangle$
  Compute $g_t := \left\langle \boldsymbol{x}^{(t)} - \boldsymbol{s}^{(t)}, \boldsymbol{r}^{(t)} \right\rangle$
  if $g_t \leq \epsilon$ **then return** $\boldsymbol{x}^{(t)}$
  Let $\gamma = \frac{2}{2+t}$ (or do line-search)
  Update $\boldsymbol{x}^{(t+1)} := (1-\gamma)\boldsymbol{x}^{(t)} + \gamma \boldsymbol{s}^{(t)}$
end for



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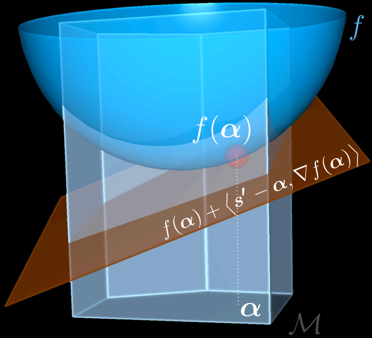

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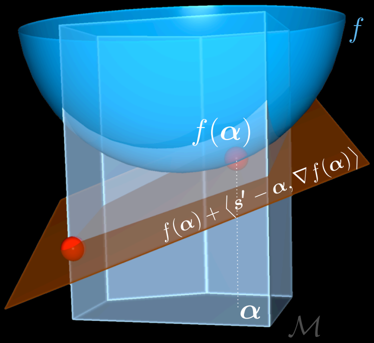

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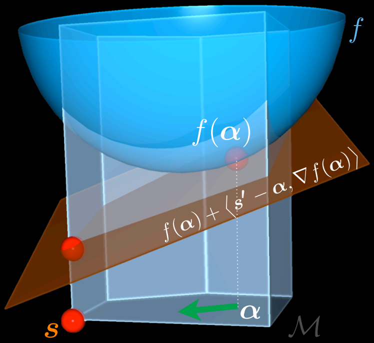

SP-FW

Algorithm Saddle point FW algorithm

Let $\boldsymbol{z}^{(0)} = (\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}) \in \mathcal{X} \times \mathcal{Y}$
for $t = 0 \dots T$ **do**
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- One can define FW extension with **away** step.
- $\gamma_t = \frac{1}{1+t} \Rightarrow \boldsymbol{z}^{(t)} = \frac{1}{t} \sum_{i=0}^{t} \boldsymbol{s}^{(i)}$.
- $(\gamma_t = \frac{1}{1+t})$ + Bilinear objective $\leftrightarrow$ *fictitious play algorithm.*

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Same main property as FW:

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When constraints set is a "complicated" ***structured*** polytope projections can be ***hard*** whereas LMO might be ***tractable***.

Theoretical contribution

SP extension of FW with *away step*:

Convergence: $\boxed{\begin{array}{c}\textbf{\textit{Linear}} \text{ rate with } \textbf{\textit{adaptive}} \text{ step size.} \\ \textbf{\textit{Sublinear}} \text{ rate with } \textbf{\textit{universal}} \text{ step size.}\end{array}}$

[3] J. Hammond. "Solving asymmetric variational inequality problems and systems of equations with generalized nonlinear programming algorithms". PhD thesis. MIT, 1984.

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- Similar hypothesis as AFW for linear convergence:
  1. Strong convexity and smoothness of the function.
  2. $\mathcal{X}$ and $\mathcal{Y}$ polytopes.

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SP extension of FW with *away step*:

Convergence: $\boxed{\begin{array}{c}\textbf{\textit{Linear}}\text{ rate with }\textbf{\textit{adaptive}}\text{ step size.}\\ \textbf{\textit{Sublinear}}\text{ rate with }\textbf{\textit{universal}}\text{ step size.}\end{array}}$

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- Proof use recent advances on AFW.
- Partially answering a **30 years old conjecture**[3].

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Difficulties for saddle point

Usual **descent Lemma**:

$$h_{t+1} \leq h_t - \underbrace{\gamma_t g_t}_{\geq 0} + \gamma_t^2 \frac{L\|\boldsymbol{d}^{(t)}\|^2}{2}$$

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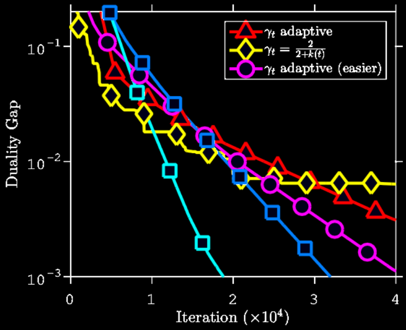
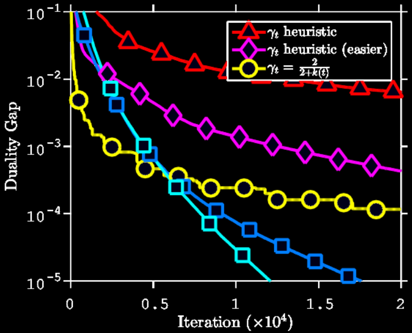
For saddle point problem the Lipschitz gradient property gives

$$\mathcal{L}_{t+1} - \mathcal{L}^* \leq \mathcal{L}_t - \mathcal{L}^* - \underbrace{\gamma_t \left(g_t^{(x)} - g_t^{(y)} \right)}_{\text{arbitrary sign}} + \gamma_t^2 \frac{L\|\boldsymbol{d}^{(t)}\|^2}{2}.$$

- Cannot control the oscillation of the sequence.
- Must introduce other quantities to establish convergence.

Toy experiments





SP-AFW on a toy example $d = 30$. with theoretical step-size $\gamma_t = \frac{\nu}{C} g_t$.

$C = 2LD^2$

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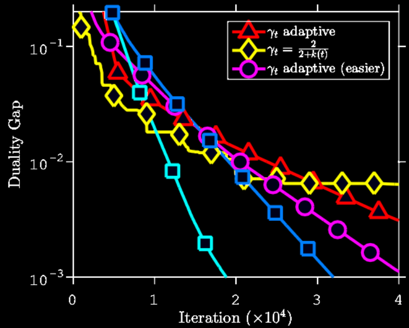

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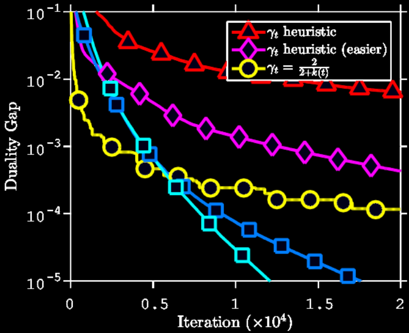

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- Rich interplay tapping into this game theory literature.

Thank You !

Slides available on `www.di.ens.fr/~gidel`.