Frank-Wolfe Algorithms for Saddle Point problems



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Overview

- Frank-Wolfe algorithm (FW) gained in popularity in the last couple of years.
- Main advantage: FW only needs LMO.
- Extend FW properties to solve saddle point problem[1].
- **Straightforward** extension but **Non trivial** analysis.

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Question for the audience: Call for application

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Saddle point and link with variational inequalities

Let $\mathcal{L} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, where $\mathcal{X}$ and $\mathcal{Y}$ are convex and compact.

$$\boxed{\text{Saddle point problem:} \quad \text{solve } \min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{y} \in \mathcal{Y}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y})}$$

A solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is called a ***Saddle Point***.

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- **Sufficient *condition***: ***Global solution*** if $\mathcal{L}$ *convex-concave.* $\forall (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$

$$\boldsymbol{x}' \mapsto \mathcal{L}(\boldsymbol{x}', \boldsymbol{y}) \text{ is convex} \quad \text{and} \quad \boldsymbol{y}' \mapsto \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}') \text{ is concave.}$$

Motivations: games and robust learning

- ***Zero-sum games with two players:***

$$\min_{\boldsymbol{x}\in\Delta(I)} \max_{\boldsymbol{y}\in\Delta(J)} \boldsymbol{x}^\top M \boldsymbol{y}$$

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- ***Robust learning:***[2] We want to learn

$$\min_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \ell\left(f_\theta(x_i), y_i\right) + \lambda\Omega(\theta)$$

with an uncertainty regarding the data:

$$\min_{\theta\in\Theta} \max_{w\in\Delta_n} \sum_{i=1}^{n} \omega_i \ell\left(f_\theta(x_i), y_i\right) + \lambda\Omega(\theta)$$

Minimize the **worst case** $\rightarrow$ gives **robustness**

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Problem with Hard projection

The ***structured SVM:***

$$\min_{\omega \in \mathbb{R}^d} \lambda \Omega(\omega) + \frac{1}{n} \sum_{i=1}^{n} \underbrace{\max_{y \in \mathcal{Y}_i} \left(L_i(y) - \langle \omega, \phi_i(y) \rangle \right)}_{\text{structured empirical loss}}$$

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Difficult to project when:

- ***Structured sparsity*** norm (group lasso norm).

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Difficult to project when:

- ***Structured sparsity*** norm (group lasso norm).
- The output $\mathcal{Y}$ is structured: ***exponential*** size.

Standard approaches in literature

- Projected gradient algorithm.

$$\boldsymbol{x}^{(t+1)} = P_{\mathcal{X}}(\boldsymbol{x}^{(t)} - \eta \nabla_x \mathcal{L}(\boldsymbol{x}^{(t)}, \boldsymbol{y}^{(t)}))$$
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- Projected extra-gradient[3].

$$\bar{\boldsymbol{x}}^{(t+1)} = P_{\mathcal{X}}(\boldsymbol{x}^{(t)} - \eta \nabla_x \mathcal{L}(\boldsymbol{x}^{(t)}, \boldsymbol{y}^{(t)}))$$
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Intuition: ***lookahead*** *move: look at what your opponent* ***would*** *do before deciding your move.*

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Prevents oscillations for non strongly convex objective.

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Standard approaches in literature

- Gradient method works for non-smooth optimization, but

$$\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{x}^{(t)},\boldsymbol{y}^{(t)}\right)\underset{T\to\infty}{\longrightarrow}(\boldsymbol{x}^*,\boldsymbol{y}^*)$$

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Even when projections are expensive:

$\boxed{\text{Can use LMO to compute approximate projections}^{4}.}$

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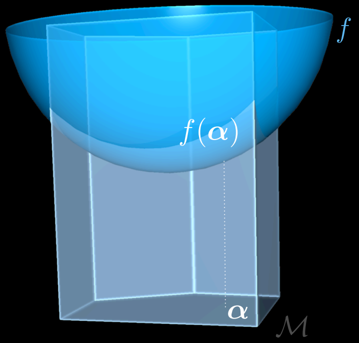
The FW algorithm

Algorithm Frank-Wolfe algorithm

1: Let $\boldsymbol{x}^{(0)} \in \mathcal{X}$
2: **for** $t = 0 \dots T$ **do**
3: Compute $\boldsymbol{r}^{(t)} = \nabla f(\boldsymbol{x}^{(t)})$
4: Compute $\boldsymbol{s}^{(t)} \in \underset{\boldsymbol{s} \in \mathcal{X}}{\operatorname{argmin}} \left\langle \boldsymbol{s}, \boldsymbol{r}^{(t)} \right\rangle$
5: Compute $g_t := \left\langle \boldsymbol{x}^{(t)} - \boldsymbol{s}^{(t)}, \boldsymbol{r}^{(t)} \right\rangle$
6: **if** $g_t \leq \epsilon$ **then return** $\boldsymbol{x}^{(t)}$
7: Let $\gamma = \frac{2}{2+t}$ (or do line-search)
8: Update $\boldsymbol{x}^{(t+1)} := (1-\gamma)\boldsymbol{x}^{(t)} + \gamma \boldsymbol{s}^{(t)}$
9: **end for**



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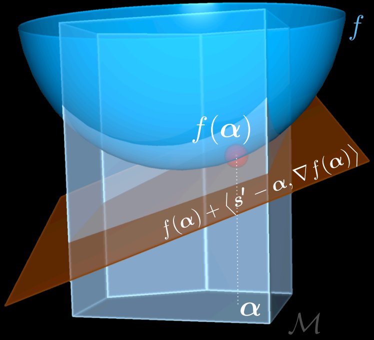

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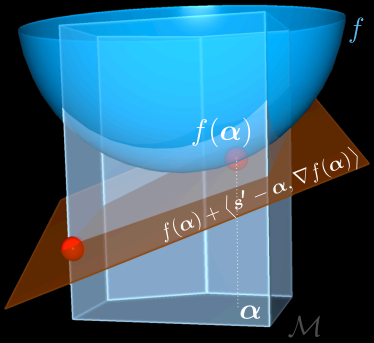

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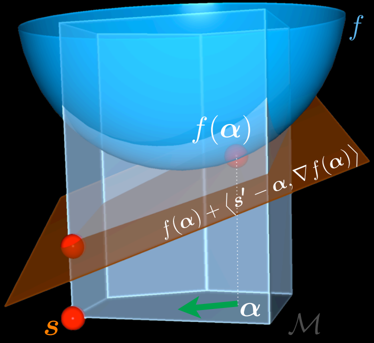

SP-FW

Algorithm Saddle point FW algorithm

for $t = 0 \dots T$ **do**

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- One can define FW extension with **away** step.

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[5] J. Hammond. “Solving asymmetric variational inequality problems and systems of equations with generalized nonlinear programming algorithms”. PhD thesis. MIT, 1984.

SP-FW

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- $\left(\gamma_t = \frac{1}{1+t}\right)$ + Bilinear objective $\leftrightarrow$ *fictitious play algorithm.*[5]

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Same main property as FW:

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When constraint set is a "complicated" ***structured*** polytope: projection is ***difficult*** whereas LMO is ***tractable***.

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Similar hypothesis as AFW:

- $\mathcal{L}$ is L-**smooth** and μ-**strongly convex-concave**.
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- Additional assumption on **bilinearity**:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x}) + \boldsymbol{x}^\top M \boldsymbol{y} - h(\boldsymbol{y})$$

Roughly, $\|M\|$ smaller than the strong convexity constant.

$$\boxed{\nu := \frac{1}{2} - \frac{\sqrt{2}\|M\|}{\mu} \frac{D}{\delta} > 0}$$

$D := \max\{\mathrm{diam}(\mathcal{X}), \mathrm{diam}(\mathcal{Y})\}$, $\delta := \min\{PWidth(\mathcal{X}), PWidth(\mathcal{Y})\}$

Theoretical contribution

SP extension of FW with *away step*[6]:

Linear rate with ***adaptive*** step size $\gamma_t := \frac{\nu}{LD^2} g_t$.

$$\min_{s \leq t} g_s \leq O(1) \left(1 - \nu^2 \frac{\delta^2}{D^2} \frac{\mu}{2L}\right)^{k(t)}$$

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- Partially answering a **30 years old conjecture**[8].
  - strongly monotone obj with step size $\frac{1}{t+1}$ over polytope.

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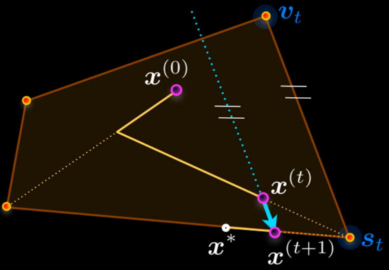
Growth Condition : Pairwise Frank Wolfe Gap



- $s_t := \arg\min_{s \in \mathcal{X}} \langle \nabla f(x^{(t)}), s \rangle.$
- $v_t := \arg\max_{v \in \mathcal{S}^{(t)}} \langle \nabla f(x^{(t)}), s \rangle$

$$\boxed{g_t^{\mathrm{PW}} := \left\langle \nabla f(x^{(t)}), v_t - s_t \right\rangle}$$

Growth Condition

Key quantity, **independent** of any algorithm[9]:

- If $\mathcal{X}$ is a polytope and f strongly convex,

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- Can extend this growth condition to SP.

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Difficulties for saddle point

Usual **descent Lemma**:

$$h_{t+1} \leq h_t - \underbrace{\gamma_t g_t}_{\geq 0} + \gamma_t^2 \frac{L\|\boldsymbol{d}^{(t)}\|^2}{2}$$

With γ_t small enough the sequence decreases.

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For saddle point problem the Lipschitz gradient property gives

$$\mathcal{L}_{t+1} - \mathcal{L}^* \leq \mathcal{L}_t - \mathcal{L}^* - \underbrace{\gamma_t \left(g_t^{(x)} - g_t^{(y)} \right)}_{\text{arbitrary sign}} + \gamma_t^2 \frac{L\|\boldsymbol{d}^{(t)}\|^2}{2}.$$

- Cannot control the oscillation of the sequence.
- Must introduce other quantities to establish convergence.

Difficulties for saddle point

Standard merit functions: *primal + dual gaps*

$$h_t := \max_{\boldsymbol{y} \in \mathcal{Y}} \mathcal{L}(\boldsymbol{x}^{(t)}, \boldsymbol{y}) - \min_{\boldsymbol{x} \in \mathcal{X}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}^{(t)}) \geq 0.$$

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We have,

$$0 \leq w_t \leq h_t \leq g_t$$

In general, w_t can be zero even if we have not reached a solution. But for strongly convex-concave function[10]

$$h_t \leq Cte\sqrt{w_t}$$

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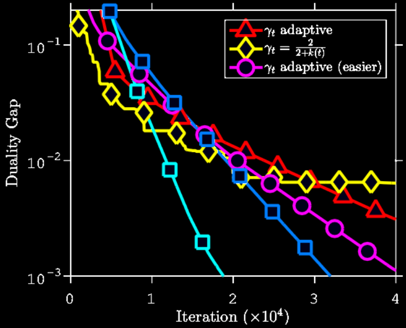
Toy Experiments

- SP-AFW with theoretical step-size.



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- $d = 30$
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Toy Experiments

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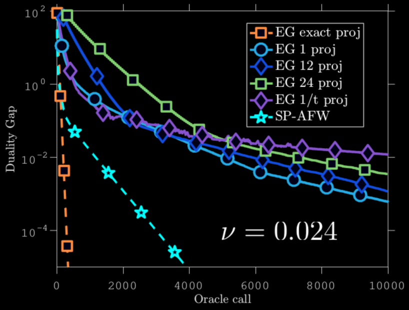

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EG : [He & Harchaoui NIPS 2015]

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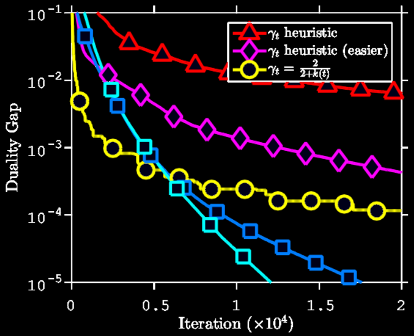

Heuristic step-size.

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Recall: theoretical step-size

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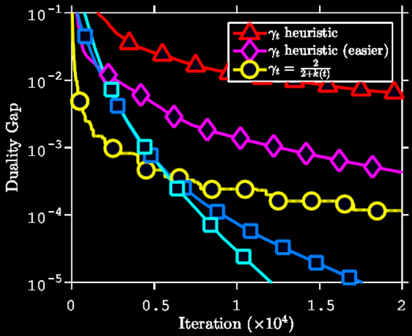

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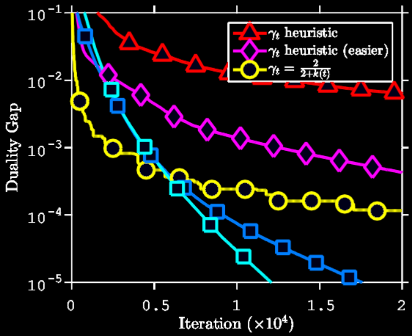

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- With a bilinear objective this algorithm is ***highly related*** to the *fictitious play algorithm.*
- Rich interplay tapping into this game theory literature.
- Still many theoretical opened questions.

  ↳ Karlin's conjecture.[11]

  ↳ Convergence without assumption on the bilinearity.

[11] Samuel Karlin. *Mathematical methods and theory in games, programming and economics.* 1960.

Thank You !

Slides available on `www.di.ens.fr/~gidel`.

Problems with difficult projection

University game:

1. Game between two universities (A and B).
2. Admitting d students and have to assign pairs of students into dorms.
3. The game has a payoff matrix M belonging to $\mathbb{R}^{(d(d-1)/2)^2}$.
4. $M_{ij,kl}$ is the expected tuition that B gets (or A gives up) if A pairs student i with j and B pairs student k with l.
5. Here the actions are both in the *marginal polytope* of all perfect *unipartite matchings*.

Hard to project on this polytope whereas the LMO can be solved efficiently with the blossom algorithm[12].

[12] J. Edmonds. "Paths, trees and flowers". In: *Canadian Journal of Mathematics* (1965).

Experiments

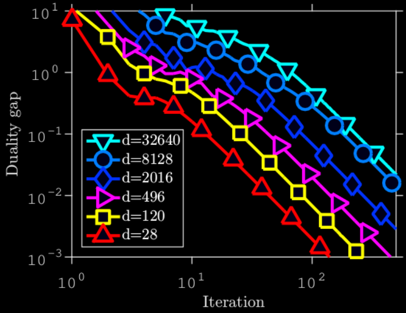

Figure: SP-FW on the University game.

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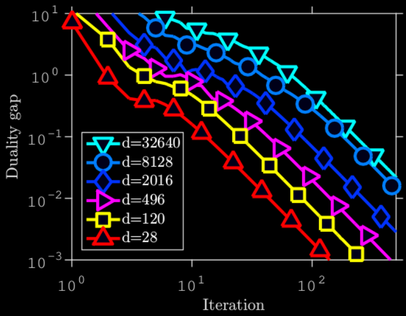

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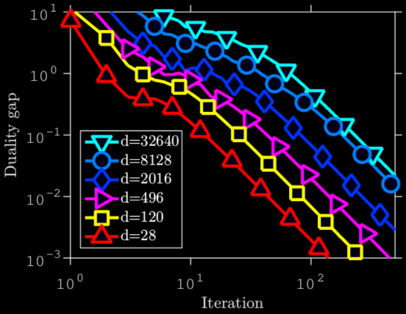

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- Scale well with dimension.